3. Definition of some "local" cardinal functions. Elementary inequalities. Counterexamples.

The *character* of X at a point $x \in X$ is $\chi(X,x) = \{\min|\mathcal{B}| : \mathcal{B}$ is a base of neighborhoods of x in $X\}$. The *character* of X is $\chi(X) = \sup\{\chi(x,X) : x \in X\}$. If $\chi(X) = \omega$ then X is called *first-countable*.

The *pseudocharacter* of X at a point $x \in X$ is $\psi(X,x) = \{\min|\mathcal{B}| : \mathcal{B}$ is a family of open sets in X such that $\cap\mathcal{B} = \{x\}\}$. The *pseudocharacter* of X is $\psi(X) = \sup\{\psi(x,X) : x \in X\}$.

The *tightness* of X at a point $x \in X$ is $\chi(X,x) = \{\min\tau :$ whenever $x \in \overline{A}$ there is $B \subset A$ such that $x \in \overline{B}$ and $|B| \leq \tau\}$. The *tightness* of X is $t(X) = \sup\{t(x,X) : x \in X\}$.

X has *countable fan tightness* at $x \in X$ if whenever $x \in \overline{A_n}$ for all $n \in \omega$ one can pick finite $F_n \subset A_n$ so that $x \in \overline{\cup_{n\in\omega} F_n}$. X has *countable fan tightness* if it has countable fan tightness at each point.

X has *countable strong fan tightness* at $x \in X$ if whenever $x \in \overline{A_n}$ for all $n \in \omega$ one can pick $p_n \in A_n$ so that $x \in \overline{\{p_n : n \in \omega\}}$. X has *countable strong fan tightness* if it has countable strong fan tightness at each point.

X is *Fréchet* if whenever $x \in \overline{A}$ there is a sequence of points of A converging to x. X is *sequential* if whenever A is not closed there is a sequence of points of A converging to some point in $\overline{A} \setminus A$.

(1) $\psi(X) \leq \min\{\chi(X), |X|\}$
(2) $t(X) \leq \min\{\chi(X), hd(X)\}$
(3) $t(X) = \omega \Leftarrow$ sequential $\Leftarrow$ Fréchet $\Leftarrow$ first-countable
(4) $t(X) = \omega \Leftarrow$ countable fan tightness $\Leftarrow$ countable strong fan tightness $\Leftarrow$ first-countable
(5) The following space V_ω is called *the countable fan*[1]: $V_\omega = \omega \times \omega \cup \{p\}$ where the points of $\omega \times \omega$ are isolated while a basic neighborhood of p takes the form $U_f = \{p\} \cup \{(m,n) \in \omega \times \omega : n > f(m)\}$ where f is some function from ω to ω. Show that V_ω (is countable and thus has countable tightness but) does not have countable fan tightness. Is V_ω sequential? ...Fréchet?
(6) Give an example of a countable space with single non-isolated point which has countable fan tightness but does not have countable strong fan tightness.
(7) (difficult) Give an example of a countable space with single non-isolated point which has countable strong fan tightness but is not first-countable.
(8) Give an example of a space X with single non-isolated point such that $\psi(X) = \omega$ but $t(X) > \omega$.
(9) Give an example of a space X with single non-isolated point such that $t(X) = \omega$ but $\psi(X) > \omega$.
(10) Give an example of a countable space which is sequential but not Fréchet.
(11) Give an example of a countable space which is Fréchet but not first-countable.

[1] or sequential fan, or ω-Fréchet-Urysohn fan