0. “Warming-up” exercises: 2^ω and similar toys

A space is called zero-dimensional if it has a base consisting of clopen sets.

Let $\mathcal{A}$ and $\mathcal{B}$ be two families of sets. Say that $\mathcal{A}$ refines $\mathcal{B}$ if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subset B$.

Partition = cover by pairwise disjoint sets.

- If X is zero-dimensional, then X has a base $\mathcal{B}$ such that $\mathcal{B}$ consists of clopen sets and $|\mathcal{B}| = w(X)$. In particular, every zero-dimensional second countable space has a countable base consisting of clopen sets.
- Prove that every zero-dimensional space X can be embedded in 2^τ where $\tau = w(X)$. (“Can be embedded” means that 2^τ contains a subspace homeomorphic to X; hint: instead of 2^τ, consider $2^{\mathcal{B}}$ where $\mathcal{B}$ is as above.)
- Show that 2^τ can be represented as the pairwise disjoint union of 2^τ many subsets each of which homeomorphic to 2^τ. Hint: note that $2^\tau \cong 2^\tau \times 2^\tau$.
- Show that if a zero-dimensional space X has a countable base of neighborhoods of $x \in X$, then X can be represented as “a point plus a sequence of pairwise disjoint clopen sets converging to this point”. Hint: let $\{U_n : n \in \omega\}$ be a sequence of clopen neighborhoods of x forming a base at this point and such that $U_{n+1} \subset U_n$ for all n. Put $K_0 = X \setminus U_0$ and $K_n = U_{n-1} \setminus U_n$ for $n \geq 1$.
- Let X be a second countable zero-dimensional space. Prove that 2^ω can be represented as the union of $\mathfrak{c}$ many pairwise disjoint subsets each of which is homeomorphic to X. Hint: enumerate the points of 2^ω on type $\mathfrak{c}$ and use two previous observations.
- Let X be a zero-dimensional space of weight τ. Is it true that 2^τ can be represented as the union of 2^τ many pairwise disjoint subsets each of which is homeomorphic to X?
- Let X be a zero-dimensional compact metrizable space, and let $\varepsilon > 0$. Show that X can be partitioned into a finite family of (pairwise disjoint) clopen sets of diameter $\leq \varepsilon$.
- Show that every zero-dimensional compact metrizable space without isolated points is homeomorphic to 2^ω. Hint: use the previous with $\varepsilon_n \to 0$ (note that the number of elements of the partition can be made a power of 2).
- Show that every countable, first countable space without isolated points is homeomorphic to $\mathbb{Q}$ (the space of rational numbers).
- Show that for an open cover $\mathcal{U}$ of a zero-dimensional Lindelöf space X, there is a (of course countable) partition $\mathcal{P}$ of X into clopen sets such that $\mathcal{P}$ refines $\mathcal{U}$.
- ([Engelking 4.3.9], The Cantor Theorem) A metric space is complete if for every sequence $F_1 \supset F_2 \supset ...$ of nonempty closed sets with $\lim\limits_{n\to\infty} \operatorname{diam}(F_n) = 0$, $\bigcap\limits_{n=1}^{\infty} F_n \neq \emptyset$.
- Show that every completely metrizable, separable zero dimensional nowhere locally compact space is homeomorphic to ω^ω. (In particular $\mathbb{P} \cong \omega^\omega$ where $\mathbb{P}$ is the space of irrational numbers.) Hint: use two previous notes.