8. Introduction to C_p and $C_p(X, 2)$

If X and Y are spaces, then Y^X is considered with the Tychonoff product topology. Points of Y^X are treated as functions from X to Y. The subspace of Y^X consisting of all continuous functions is denoted by $C_p(X, Y)$. The basic open sets in $C_p(X, Y)$ are of the form

$$W(x_1, ...x_k, U_1, ...U_k) = \{g \in C_p(X, Y) : g(x_1) \in U_1, ..., g(x_k) \in U_k\}$$

where $k \in \mathbb{N}$, $x_1, ..., x_k \in X$ and $U_1, ...U_k$ are open sets in Y.

Proposition 1. ([Arh], Proposition 0.3.1) *If $\mathcal{B}$ is a base of Y, then* $\{W(x_1, ...x_k, U_1, ...U_k) : x_1, ..., x_k \in X,\ \ U_1, ...U_k \in \mathcal{B},\ \ k \in \mathbb{N}\}$ *is a base of* $C_p(X, Y)$.

In the case $Y = \mathbb{R}$, $C_p(X, \mathbb{R})$ is denoted simply by $C_p(X)$. The basic open sets in $C_p(X)$ may be taken of the form

$$W(f, x_1, ..., x_k, \varepsilon) = \{g \in C_p(X) : |g(x_i) - f(x_i)| < \varepsilon \ \text{ for } \ i = 1, ..., k\}$$

where $f \in C_p(X)$, $x_1, ..., x_k \in X$ and $\varepsilon > 0$.

In the case $Y = 2$ (or $Y =$ a discrete space of larger cardinality), the basic open sets also take some specially simple form. Let $\mathcal{F}_X$ be the set of all finite functions from X to 2 (that is, functions whose domains are finite subsets of X.) For $\varphi \in \mathcal{F}_X$, put

$$W(\varphi) = \{g \in C_p(X, 2) : g|_{\mathrm{dom}(\varphi)} = \varphi\}.$$

Alternatively, for $f \in C_p(X, 2)$ and a finite $A \subset X$, put

$$W(f, A) = \{g \in C_p(X, 2) : g|_A = f|_A\}.$$

Proposition 2. ([Arh], Proposition 0.3.2) *If $Y \subset Z$, then $C_p(X, Y)$ is a subspace in $C_p(X, Z)$. Moreover, if Y is closed in Z, then $C_p(X, Y)$ is closed in $C_p(X, Z)$.*

Proposition 3. ([Arh], Proposition 0.3.3) *$C_p(X, \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ is canonically homeomorphic to $\prod\{C_p(X, Y_\alpha) : \alpha \in \mathcal{A}\}$.*

Proposition 4. ([Arh], Proposition 0.3.4) *Let $X = \sum_{\oplus}\{X_\alpha : \alpha \in \mathcal{A}\}$ be a discrete topological sum. Then $C_p(X, Y)$ is canonically homeomorphic to $\prod\{C_p(X_\alpha, Y) : \alpha \in \mathcal{A}\}$.*

Proposition 5. ([Arh], Proposition 0.3.5) *$C_p(X)$ is a locally convex linear topological space over $\mathbb{R}$.*

Proposition 6. ([Arh], Proposition 0.3.6) *$C_p(X)$ is dense in $\mathbb{R}^X$.*

Say that X is *weakly zero-dimensional* if every two distinct points of X can be separated by clopen sets.

Proposition 7. *If X is weakly zero-dimensional, then for every Y, $C_p(X, Y)$ is dense in Y^X.*

- In particular, if X is weakly zero-dimensional, then $C_p(X, 2)$ is dense in 2^X, $C_p(X, \omega)$ is dense in ω^X, etc. Henceforward, we consider $C_p(X, 2)$ or $C_p(X, \omega)$ only for weakly zero-dimensional X.
- Can Proposition 7 be conversed?
- Is there a weakly zero-dimensional space which is not zero-dimensional? Equivalently: let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$, Suppose $(X, \mathcal{T}_1)$ is zero-dimensional. Must $(X, \mathcal{T}_2)$ be zero-dimensional?

Proposition 8. ([Arh], Corollary 0.3.7) *$c(C_p(X)) = \omega$ for every X.*

Proposition 9. *If Y is separable, then $c(C_p(X, Y)) = \omega$ for every weakly zero-dimensional space X.*

Proposition 10. ([AP], p. 300 of the Russian edition, No 133) *If a C.C.C. space X is paracompact, then X is Lindelöf.*

(Hint: paracompact ⇔ every open cover has a σ-discrete open refinement.)

Proposition 11. ([Arh], remark after Corolary 0.3.7) *$C_p(X)$ is paracompact iff $C_p(X)$ is Lindelöf.*

- The same is true for $C_p(X, Y)$ when X is weakly zero-dimensional and Y separable.
- Show that in general $C_p(X, Y)$ may be paracompact but not Lindelöf.
- Suppose all finite powers of Y are Lindelöf and $C_p(X, Y)$ is paracompact. Must $C_p(X, Y)$ be Lindelöf?

Proposition 12. ([Arh], Theorem 1.1.1) $|X| = \chi(C_p(X)) = w(C_p(X))$.

Proposition 13. ([Arh], Proposition 0.3.14) *Every uncountable regular cardinal is a precaliber of $C_p(X)$.*

(1) The same is true for $C_p(X, Y)$ whenever X is weakly zero-dimensional and Y separable.
(2) If X is discrete, then $C_p(X, Y) = Y^X$.
(3) If both X and Y are countable, first countable, and do not have isolated points, then X and Y are homeomorphic (to each other and to $\mathbb{Q}$, the space of rational numbers.)
(4) If X is a (infinite) zero-dimensional compact space, then $|C_p(X, 2)| = w(X)$. In particular, if X is a second countable zero-dimensional compact space, then $C_p(X, 2)$ is countable.
(5) If X is pseudocompact, then $C_p(X) = \bigcup_{n\in\omega} C_p(X, [-n, n])$ and $C_p(X, \omega) = \bigcup_{n\in\omega} C_p(X, n)$.
(6) If X is a second countable zero-dimensional compact space, then $C_p(X, \omega)$ is countable.
(7) By the way, the previous result provides an alternative proof of the special case of Hewitt-Marczewski-Pondiczery theorem: if Z is separable, then $Z^{\mathfrak{c}}$ is separable. Instead of $Z^{\mathfrak{c}}$, consider Z^K where K is some zero dimensional second countable compact space (for example, the Cantor set $K = 2^\omega$). Let D be a dense countable subspace in Z. Let $\mathcal{D}$ be the discrete topology on D. Then $C_p(K, (D, \mathcal{D}))$ can be regarded as a (dense) subspace in Z^K.
(8) If X countably infinite and compact, then $C_p(X, 2)$ and $C_p(X, \omega)$ are homeomorphic to $\mathbb{Q}$.

A space Z is *perfectly $\varkappa$-normal* if every regular closed set is a zero-set.

Proposition 14. *For every τ, $\mathbb{R}^\tau$ is perfectly $\varkappa$-normal.*

Proof: Let U be open in $\mathbb{R}^\tau$. Extract a maximal family $\mathcal{V}$ of pairwise disjoint basic open sets contained in U. Then $\mathcal{V}$ is at most countable. For every $V \in \mathcal{V}$ there is finite $K_V \subset \tau$ such that $\pi_{K_V}^{-1}(\pi_{K_V}(V)) = V$. Put $K = \cup_{V\in\mathcal{V}} K_V$. Then K is at most countable. We have $\pi_K^{-1}(\pi_K(\cup\mathcal{V})) = \cup\mathcal{V}$ and thus $\pi_K^{-1}(\pi_K(\overline{\cup\mathcal{V}})) = \overline{\cup\mathcal{V}}$. But $\cup\mathcal{V}$ is dense in U, so $\overline{\cup\mathcal{V}} = \overline{U}$. Therefore $\pi_K^{-1}(\pi_K(\overline{U})) = \overline{U}$. Now, $\pi_K(\overline{U})$ is a closed set in a second countable space, thus a zero-set. There is a continuous function $h : \mathbb{R}^K \to \mathbb{R}$ such that $\pi_K(\overline{U}) = h^{-1}(0)$. Then $\overline{U} = (h \circ \pi_K)^{-1}(0)$. $\square$

Proposition 15. *A dense subspace in a perfectly $\varkappa$-normal space is perfectly $\varkappa$-normal.*

Proposition 16. ([Arh], Proposition 0.3.16) *$C_p(X)$ is perfectly $\varkappa$-normal.*

- Two disjoint regular closed sets in a perfectly $\varkappa$-normal space can be separated.
- A closed set in a perfectly $\varkappa$-normal space does not have to be a zero set.