

4. Elementary inequalities

In this section we will make an exception and will not assume all spaces to be Tychonoff.

- (1) If X is Hausdorff, then $|X| \leq 2^{2^{d(X)}}$
- (2) The previous inequality is the best possible even for Tychonoff spaces.
- (3) X is Hausdorff, then $w(X) \leq 2^{2^{2^{d(X)}}$
- (4) (Difficult) The previous inequality is the best possible for Hausdorff spaces.
- (5) Can one write something better for nw ?
- (6) If X is regular, then $w(X) \leq 2^{d(X)}$
- (7) If X is Hausdorff, then $|X| \leq 2^{w(X)}$
- (8) Can one replace w with nw in the previous?
- (9) Can one replace w with iw (for Tychonoff spaces)?
- (10) Can one replace w with πw ?
- (11) Show that the cardinality of a space can not be bound by l alone.
- (12) nor even by l and c together
- (13) nor even by l , c , and t together (*Hint*: consider a σ -product in 2^τ .)
- (14) (By the way) Even a Σ -product in 2^τ has countable tightness.
- (15) If X is T_1 , then $|X| \leq nw(X)^{\psi(X)}$
- (16) If X is Hausdorff, then $|X| \leq d(X)^{\chi(X)}$
- (17) One can't replace Hausdorff with T_1 in the previous (there is no bound on the cardinality of separable first countable T_1 spaces.)
- (18) If X is regular, then $|X| \leq 2^{d(X)\psi(X)}$.
- (19) Regular can't be replaced with Hausdorff in the previous.
- (20) (Jones' Lemma) If X is normal and D a closed discrete subspace of X , then $2^{|D|} \leq 2^{d(X)}$
- (21) So, assuming CH, every separable normal space has countable extent.
- (22) Assume there exist a Q-set.¹ Then there is a separable normal first countable (even Moore²) space of uncountable extent. *Hint*: consider the subspace of the upper half-plane with Niemytzky topology when in the bottom line we take only a subset which is a Q-set in \mathbb{R} .³

¹That is, such a subspace $Q \subset \mathbb{R}$ that for every $A \subset Q$, A is of type F_σ in Q . Uncountable Q-sets exist, for example, assuming $MA+\neg CH$ - see more in Tall's Handbook article, p. 709-710.

²This means that there exists a sequence $(\mathcal{U}_n : n \in \omega)$ of open covers such that for every $x \in X$ and every open set $O \ni x$ there is n such that $St(\{x\}, \mathcal{U}_n) \subset O$.

³Some more examples can be found in Tall's Handbook article.