0.2 More fun problems

(some of these were already discussed)

(1) Let X be a set of cardinality $\leq \mathfrak{c}$ and let $f : X \to X$ be any mapping. Show that there is a second countable zero-dimensional topology on X with respect to which f is continuous.

(2) Can one request in addition that the mapping is open?

(3) Can one request in addition that the topology is Hausdorff?

(4) State and prove similar results for a mapping $f : X \to Y$.

(5) In particular, let X be a set of cardinality $\leq \mathfrak{c}$. Consider $f : X \to \mathbb{R}$. Is there a zero-dimensional Hausdorff topology on X with respect to which f is continuous?

(6) Can one request in addition that f (considered as a mapping from X onto $f(X)$) is open?

(7) For what $(X, \mathcal{T})$, can one construct a bijection $f : X \to X$ such that the topology $\mathcal{T} \bigvee f(\mathcal{T})$ [1] is discrete? Can one do it, for example, with the Cantor set?

[1] That is, the topology on X generated by the family of sets $\mathcal{T} \bigcup \{f(T) : T \in \mathcal{T}\}$ as by a subbase