3. Cardinal functions on products - 1

- If φ is one of the cardinal functions defined in the previous notes, then $\varphi(\prod_{\alpha\in\mathcal{A}} X_\alpha) \geq \varphi(X_\alpha)$ for every $\alpha \in \mathcal{A}$.
- If $w(X_\alpha) \leq \tau$ for all $\alpha \in \mathcal{A}$, and $|\mathcal{A}| \leq \tau$, then $w(\prod_{\alpha\in\mathcal{A}} X_\alpha) \leq \tau$.
- Is similar inequality true for $nw(X)$ and $iw(X)$?

Theorem 1. (Hewitt-Marczewski-Pondiczery, see [Eng], Theorem 2.3.15.) *If $d(X_\alpha) \leq \tau$ for all $\alpha \in \mathcal{A}$, and $|\mathcal{A}| \leq 2^\tau$, then $d(\prod_{\alpha\in\mathcal{A}} X_\alpha) \leq \tau$.*

A cardinal τ is a *caliber* of $(X, \mathcal{T})$ if for every family $\{G_\alpha : \alpha < \tau\} \subset \mathcal{T} \setminus \{\emptyset\}$, there is $A \subset \tau$ with $|A| = \tau$ such that $\cap_{\alpha\in A} G_\alpha \neq \emptyset$. A cardinal τ is a *precaliber* of $(X, \mathcal{T})$ if for every family $\{G_\alpha : \alpha < \tau\} \subset \mathcal{T} \setminus \{\emptyset\}$, there is $A \subset \tau$ with $|A| = \tau$ such that the family $\{G_\alpha : \alpha \in A\}$ is centered.[1]

$\mathrm{cal}(X) = \{\tau : \tau \text{ is a caliber of } X\}$ $\quad$ $\mathrm{precal}(X) = \{\tau : \tau \text{ is a precaliber of } X\}$

- $\mathrm{cal}(X) \subset \mathrm{precal}(X)$.
- If Y is dense in X, then $\mathrm{precal}(Y) = \mathrm{precal}(X)$.
- If X is compact, then $\mathrm{cal}(X) = \mathrm{precal}(X)$.
- If $cf(\tau) > d(X)$, then $\tau \in cal(X)$.
- In particular, ω_1 is a caliber of any separable space.

A family of sets $\mathcal{D}$ is called a Δ-*system* if there exists a set R (called the *root* of $\mathcal{D}$) such that for every two distinct $A, B \in \mathcal{D}$, $A \cap B = R$.[2]

Proposition 2. (The Δ-system lemma, see [Kunen, Theorem 1.6] for a slightly more general statement) *If $\mathcal{F}$ is a family of finite sets, and $|\mathcal{F}| = \tau$ is an uncountable regular cardinal, then there exists a Δ-system $\mathcal{D} \subset \mathcal{F}$ such that $|\mathcal{D}| = \tau$.*

Sketch of proof: First, for some $n \geq 0$, there is $\mathcal{F}' \subset \mathcal{F}$ such that $|\mathcal{F}' = \tau$ and $|A| = n$ for all $A \in \mathcal{F}'$. Consider $\mathcal{F}'$ instead of $\mathcal{F}$. The proof is by induction on n. If $n = 0$ we set $\mathcal{D} = \mathcal{F}'$. Now, let $n > 0$ and suppose the statement is true for $n-1$. There are two cases: <u>Case 1</u> There is $a \in \cup\mathcal{F}'$ such that $|\{F \in \mathcal{F}' : a \in F\}| = \tau$. Then put $\mathcal{G} = \{F \setminus \{a\} : F \in \mathcal{F}'\}$ and apply the inductive assumption. <u>Case 2</u> No a like in case 1 exists. For $a \in \cup\mathcal{F}'$, put $St^1(a) = \cup\{F \in \mathcal{F}' : a \in F\}$, ..., $St^{n+1}(a) = \cup\{F \in \mathcal{F}' : F \cap St^n(a) \neq \emptyset\}$, ..., $St^\omega(a) = \cup\{St^n(a) : n \in \omega\}$. One can check that for every a, $St^\omega(a)$ has cardinality less than τ, and that for different a, the sets $St^\omega(a)$ either coincide or do not intersect. So there are τ many pairwise disjoint sets $St^\omega(a)$. For each of those pick an element of $\mathcal{F}'$ which is a subset of it, and thus we get a Δ-system of cardinality τ with empty root. $\Box$

Proposition 3. *If for every $\alpha \in \mathcal{A}$, $\tau \in \mathrm{precal}(X_\alpha)$ (or $\tau \in \mathrm{cal}(X_\alpha)$), then $\tau \in \mathrm{precal}(\prod_{\alpha\in\mathcal{A}} X_\alpha)$ (respectively, $\tau \in \mathrm{cal}(\prod_{\alpha\in\mathcal{A}} X_\alpha)$).)*

Sketch of proof: First, prove as a lemma that the statement is true for finite products. Then, in the general case, assume w.l.o.g. that the family $\{G_\alpha : \alpha < \tau\} \subset \mathcal{T} \setminus \{\emptyset\}$ (where $\mathcal{T}$ is the topology of the entire product) consists of basic open sets: each G_α depends only on coordinates from a finite subset $A_\alpha \subset \mathcal{A}$. Then apply the Δ-system lemma together with the lemma about the finite product. $\Box$

Theorem 4. *If for every $\alpha \in \mathcal{A}$, $d(X_\alpha) \leq \kappa$, then $c(\prod_{\alpha\in\mathcal{A}} X_\alpha) \leq \kappa$.*

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[1] A family of sets is *centered* if every non-empty finite subfamily has non-empty intersection

[2] In particular, R may be empty

- A dense subspace in a product of separable spaces is CCC.
- Calculate the cardinal functions w, d, etc. for the space 2^τ (you may want to use the notation $\log\tau$ for $\min\{\kappa : 2^\kappa \geq \tau\}$.)
- Is similar conclusion true for nw, πw, iw, χ, ψ?
- If $|X_\alpha| > 1$ for all $\alpha \in \mathcal{A}$, and $|\mathcal{A}| > \tau$, then $w(\prod_{\alpha\in\mathcal{A}} X_\alpha) > \tau$ etc.
- Disprove the statement $l(X^2) = l(X)$
- Show that a dense subspace of a product may have density greater than the whole product.