

6. Coloring arguments

If S is a set and τ a cardinal then $[S]^\tau = \{A \subset X : |A| = \tau\}$.

Theorem 1. (Erdős, Rado, see [Hodel-Handbook] 2.3 or [Juhasz] A4.4) *Let κ be an infinite cardinal, and S a set of cardinality $|S| > 2^\kappa$, and let $[S]^2 = \bigcup_{\alpha < \kappa} P_\alpha$. Then there are $\alpha < \kappa$ and a subset $A \subset P_\alpha$ such that $|A| > \kappa$ and $[A]^2 \subset P_\alpha$.*

Symbolically, this is expressed as

$$(2^\kappa)^+ \rightarrow (\kappa^+)_\kappa^2 \quad \text{in particular,} \quad \mathfrak{c}^+ \rightarrow (\omega_1)_\omega^2$$

This can be interpreted as follows: if the edges of a complete graph with more than \mathfrak{c} vertices are painted in countably many colors, then there exists a monochromatic complete subgraph with uncountably many vertices.

1. PROOF OF ERDŐS-RADO THEOREM

Rather long proofs (that mimic Ramsey arguments in finite combinatorics) can be found in [Hodel-Handbook] or [Juhasz]. Here is a shorter proof from [1]. First, Hodel gives a combinatorial theorem with a “saturation” proof, and then he shows how to derive the Saturation Theorem and the Coloring Theorem from each other.

Theorem 2. (Hodel’s “First Combinatorial Theorem”, [1]) *Let κ be an infinite cardinal and S a set. For $x \in X$ and $\gamma < \kappa$, let $x \in V(\gamma, x) \subset S$. Assume that*

(S) *if $x \neq y$ then for some $\gamma < \kappa$, $y \notin V(\gamma, x)$;*

(HC) *for each $\gamma < \kappa$ and each $A \subset S$, there is $B \subset A$ such that $|B| \leq \kappa$ and*

$$A \subset \bigcup_{x \in B} V(\gamma, x)$$

Then $|S| \leq 2^\kappa$

Sketch of Proof. Construct a sequence $(S_\alpha : \alpha < \kappa^+)$ of subsets of S such that

(1) $|S_\alpha| \leq 2^\kappa$;

(2) If for each $\gamma < \kappa$, $B_\gamma \subset \bigcup_{\beta < \alpha} S_\beta$ and $|B_\gamma| \leq \kappa$, and $W \neq S$ where $W = \bigcup_{\gamma < \kappa} (\bigcup_{x \in B_\gamma} V(\gamma, x))$, then $S_\alpha \setminus W \neq \emptyset$.

Put $L = \bigcup_{\alpha < \kappa^+} S_\alpha$. Then $|L| \leq 2^\kappa$ and it remains to show that $L = S$. To do this, assume $y \in S \setminus L$. For $\gamma < \kappa$ put $A_\gamma = \{x \in L : y \notin V(\gamma, x)\}$. By (S), $L = \bigcup_{\gamma < \kappa} A_\gamma$. By (HC), there are $B_\gamma \subset A_\gamma$ such that $|B_\gamma| \leq \kappa$ and $A_\gamma \subset \bigcup_{x \in B_\gamma} V(\gamma, x)$. Put $W = \bigcup_{\gamma < \kappa} (\bigcup_{x \in B_\gamma} V(\gamma, x))$. Then $L \subset W$ and $y \notin W$. There is $\alpha < \kappa^+$ such that $\bigcup_{\gamma < \kappa} B_\gamma \subset \bigcup_{\beta < \alpha} S_\beta$. By (2), $S_\alpha \setminus W \neq \emptyset$ which contradicts $L \subset W$. \square

Hodel’s First Theorem \Rightarrow Erdős-Rado Theorem Let $V(\gamma, x) = \{x\} \cup \{y \in S \setminus \{x\} : \{x, y\} \notin P_\gamma\}$. Then (S) holds, but $|S| > 2^\kappa$, so (HC) fails which is witnessed by some $A \subset S$ and $\gamma < \kappa$. Then one can pick $x_\alpha \in A$ for all $\alpha < \kappa^+$ so that $x_\alpha \notin \bigcup_{\beta < \alpha} V(\gamma, x_\beta)$. Put $B = \{x_\alpha : \alpha < \kappa^+\}$. Then $|B| > \kappa$ and $B \subset P_\gamma$. \square

Erdős-Rado Theorem \Rightarrow Hodel’s First Theorem Suppose S , κ and the sets $V(\gamma, x)$ satisfy (S) and (HC), but $|S| > 2^\kappa$. Well order S by \preceq and put $P_\gamma = \{\{x, y\} : x \prec y \text{ and } y \notin V(\gamma, x), \text{ or } y \prec x\}$. By (S), $[S]^2 \subset \bigcup_{\gamma < \kappa} P_\gamma$ so there is $A \subset S$ such that $|A| > \kappa$ and $[A]^2 \subset P_\gamma$ for some $\gamma < \kappa$. We may assume that A , with respect to \preceq , is order-isomorphic to κ^+ . Take $B \subset A$ as in (HC). Then there is $y \in A$ such that $x \prec y$ for all $x \in B$. On the one hand, by the definition of B , $y \in V(\gamma, x)$ for some $x \in B$. On the other hand, this is impossible by the definition of P_γ . \square

2. APPLICATIONS OF ERDÖS-RADO THEOREM

Theorem 3. (Hajnal-Juhász) *If X is (Hausdorff) first-countable and C.C.C. then $|X| \leq \mathfrak{c}$.*

Sketch of a Ramseyan proof: Assume $|X| > \mathfrak{c}$. For each $x \in X$, fix a base of neighborhoods $\{B_n(x) : n \in \omega\}$. For $n \in \omega$, put $P_n = \{\{x, y\} \in [X]^2 : B_n(x) \cap B_n(y) = \emptyset\}$. Now arrive to a contradiction with C.C.C. \square

Theorem 4. (Hajnal-Juhász, see [Hodel], 4.9) *If X is T_1 , and $hc(X) = \psi(X) = \omega$ then $|X| \leq \mathfrak{c}$.*

Sketch of proof: Like the previous, but we assume only $\bigcap\{B_n(x) : n \in \omega\} = \{x\}$ and put $P_n = \{\{x, y\} \in [X]^2 : B_n(x) \not\ni y \text{ and } B_n(y) \not\ni x\}$. \square

A space X is called *star-Lindelöf* if for every open cover \mathcal{U} there is a countable $C \subset X$ such that $St(C, \mathcal{U}) = X$. All separable spaces, and all spaces of countable extent (thus in particular all Lindelöf spaces and all countably compact spaces) are star-Lindelöf. More generally, $st-L(X) = \min\{\tau : \text{for every open cover } \mathcal{U} \text{ there is } C \subset X \text{ such that } |C| \leq \tau \text{ and } St(C, \mathcal{U}) = X\}$ is called the *star-Lindelöf number* of X .¹

Theorem 5. (Bonanazinga) *If X is a Hausdorff first-countable star-Lindelöf space, then $e(X) \leq \mathfrak{c}$.*

Sketch of proof: Suppose Z be a closed discrete subspace of X such that $|Z| > \mathfrak{c}$. For every $x \in Z$, let $\{B_n(x) : n \in \omega\}$ be a base of neighborhoods in X . For $n \in \omega$, put $P_n = \{\{x, y\} \in [Z]^2 : B_n(x) \cap B_n(y) = \emptyset\}$ and $\mathcal{U}_n = \{B_n(x) : x \in Z\} \cup \{X \setminus Z\}$ (then \mathcal{U}_n is an open cover of X). Now arrive to a contradiction with star-Lindelöfness. \square

- (1) More generally $e(X) \leq 2^{\chi(X) \cdot st-L(X)}$ for Hausdorff X .
- (2) (If we drop the restriction on the character in the previous) the extent of a star-Lindelöf Tychonoff space can be arbitrarily large. (See the next section.)
- (3) (By the way) the extent of a normal star-Lindelöf space is at most \mathfrak{c} .
- (4) There is no bound on $|X|$ using only $\chi(X)$ and $st-L(X)$.

¹Or the *weak extent* of X , denoted by $we(X)$.

3. MORE APPLICATIONS OF HODEL'S FIRST THEOREM

Theorem 6. (de Groot) $|X| \leq 2^{hl(X)}$ for Hausdorff X .

Sketch of proof: Note that $\psi(X) \leq hl(X)$ for Hausdorff X . \square

The *diagonal* of X is $\Delta = \{ \langle x, x \rangle \in X^2 : x \in X \}$. The *diagonal number* of a T_1 space X is $\Delta(X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a family of open sets in } X^2 \text{ such that } \bigcap \mathcal{U} = \Delta\}$. If $\Delta(X) = \omega$, we say that X has a G_δ -*diagonal*.

Lemma 7. $\Delta(X) = \min\{\kappa : \text{there is a sequence } (\mathcal{C}_\alpha : \alpha < \kappa) \text{ of open covers of } X \text{ such that for every } x \in X, \{x\} = \bigcap_{\alpha < \kappa} St(x, \mathcal{C}_\alpha)\}$.

Lemma 8. *Let X be a T_1 space and κ an ordinal. Then the following conditions are equivalent:*

- (1) $e(X) \leq \kappa$
- (2) *For every open cover \mathcal{C} of X and every $A \subset X$ there is $B \subset A$ such that $|B| \leq \kappa$ and $A \subset St(B, \mathcal{C})$.*

Theorem 9. (Ginsburg-Woods) $|X| \leq 2^{e(X)\Delta(X)}$ for a T_1 space X .

Sketch of proof: Use Lemma 7 to get (S) and Lemma 8 to get (HC). \square

REFERENCES

- [1] R. E. Hodel, *Combinatorial set theory and cardinal functions inequalities*, Proc. Amer. Math. Soc. **111** (1991) No 2, 567-575.