5. “Saturation” arguments

Theorem 1. (Arhangelskii) *If X is Hausdorff, Lindelöf and first-countable, then $|X| \leq \mathfrak{c}$.*

Sketch of proof:[1] For every $x \in X$, let $\mathcal{B}_x$ be a countable base of neighborhoods of x. Construct increasing chains $(H_\alpha : \alpha < \omega_1)$ of closed subsets of X and $(\mathcal{B}(\alpha) : \alpha < \omega_1)$ of families of open sets in X such that:

(1) $|H_\alpha| \leq \mathfrak{c}$.

(2) $\mathcal{B}(\alpha) = \bigcup\{\mathcal{B}_x : x \in \bigcup\{H_\beta : \beta < \alpha\}\}$.

(3) If $\mathcal{W}$ is a countable subfamily of $\mathcal{B}(\alpha)$ and $\bigcup\mathcal{W} \neq X$ then $H_\alpha \setminus \bigcup\mathcal{W} \neq \emptyset$.

To start, pick arbitrary $y_0 \in X$ and put $H_0 = \{y_0\}$ and $\mathcal{B}(0) = \emptyset$.

As soon as $0 < \beta < \omega_1$ and H_α and $\mathcal{B}(\alpha)$ have been constructed for all $\alpha < \beta$, define $\mathcal{B}(\beta)$ according to (2) and put $\mathcal{E}_\beta = \{\mathcal{W} \subset \mathcal{B}(\beta) : |\mathcal{W}| \leq \omega$ and $\bigcup\mathcal{W} \neq X\}$. For each $\mathcal{W} \in \mathcal{E}_\beta$ pick a point $\varphi(\mathcal{W}) \in X \setminus \bigcup\mathcal{W}$. Put $H_\beta = \overline{\{\varphi(\mathcal{W}) : \mathcal{W} \in \mathcal{E}_\beta\} \cup \bigcup\{H_\alpha : \alpha < \beta\}}$. Then $|H_\beta| \leq \mathfrak{c}$.

As soon as H_α and $\mathcal{B}(\alpha)$ have been constructed for all α, put $H = \bigcup\{H_\alpha : \alpha < \omega_1$. Then $H \leq \mathfrak{c}$.

Claim 1. H is closed in X. (.... use first-countability to verify this...)

Claim 2. $H = X$ (... prove it....) □

(1) The previous theorem has the following generalization: $|X| \leq 2^{l(X)\cdot\chi(X)}$ for every Hausdorff X.

If $\mathcal{B}$ is a family of subsets of a set X, and $x \in X$, we put $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. A *pseudobase* of X is a family $\mathcal{B}$ of open sets such that for every $x \in X$, $\bigcap\mathcal{B}_x = \{x\}$. A space X is *countably compact* if every countable open cover of X contains a finite subcover (equivalently, if every infinite set has an accumulation point). A space is *pseudocompact* if every continuous real-valued function defined on X is bounded. (See [Engelking], Section 3.10).

Theorem 2. (Arhangelskii, Proizvolov) *If X is countably compact, Y a dense subspace of X, and $\mathcal{B}$ a pseudobase of X such that $|\mathcal{B}_y| \leq \omega$ for all $y \in Y$, then X is compact and $\mathcal{B}$ is a countable base of X.*

Sketch of proof:[2] Construct increasing chains $(M_\alpha : \alpha < \omega)$ of countable subsets of Y, and $(\mathcal{B}_\alpha : \alpha < \omega)$ of subfamilies of of $\mathcal{B}$ as follows:

Pick $y_0 \in Y$ arbitrarily and put $M_0 = \{y_0\}$.

As soon as M_α has been defined for some $\alpha < \omega$, put $\mathcal{B}(\alpha) = \bigcup\{\mathcal{B}_y : y \in M_\alpha\}$ and $\mathcal{E}_\alpha = \{\mathcal{F} \subset \mathcal{B}_\alpha : |\mathcal{F}| < \omega$ and $Y \setminus \bigcup\mathcal{F} \neq \emptyset\}$. For each $\mathcal{F} \in \mathcal{E}_\alpha$ pick $\varphi(\mathcal{F}) \in Y \setminus \bigcup\mathcal{F}$. Put $\beta = \alpha + 1$ and $M_\beta = M_\alpha \cup \{\varphi(\mathcal{F}) : \mathcal{F} \in \mathcal{E}_\alpha\}$. Then $\mathcal{E}_\alpha$ and M_β are countable and we can continue.

As soon as M_α and $\mathcal{B}_\alpha$ have been defined for all α, put $M = \bigcup\{M_\alpha : \alpha < \omega\}$ and $\tilde{\mathcal{B}} = \bigcup\{\mathcal{B}_\alpha : \alpha < \omega\}$. Then M and $\tilde{\mathcal{B}}$ are countable.

Claim 1. M is dense in X. (.... prove this!....)

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[1] This argument is known as the “Pol’s proof of Arhangelskii Theorem”

[2] This argument is a variation of a proof of a slightly weaker result attributed to M.-E. Rudin in a paper of Corson and Michael; Rudin’s argument and Pol’s proof of Arhangelskii theorem are historically first instances of saturation method

Therefore, $\tilde{\mathcal{B}} = \mathcal{B}$, so $\mathcal{B}$ is a countable pseudobase of a countably compact space X. It remains to show that $\mathcal{B}$ is a base (.... do it!....)[3] $\square$

(1) The previous theorem has the following variation. Say that $Y \subset X$ is relatively countably compact in X if every infinite subset of Y has an accumulation point in X. A space with a dense relatively countably compact subspace is called *densely countably compact*. Every densely countably compact space with a point-countable base is compact (and the base is countable).

(2) Countable compactness can not be replaced in Theorem 2 with pseudocompactness: a Ψ-space has a point-countable pseudobase.[4]

(3) Theorem 2 has a generalization that uses the cardinal function called *index of compactness*: $ic(X) = \min\{\tau$: there is an open cover $\mathcal{U}$ of X such that $|\mathcal{U}| \leq \tau$ and $\mathcal{U}$ does not contain a finite cover of X.[5] State and prove this generalization.

Theorem 3. (Hajnal-Juhasz) *If X is (Hausdorff) first-countable and C.C.C. then $|X| \leq \mathfrak{c}$.*

Sketch of proof:[6]. For every $x \in X$, let $\mathcal{B}_x$ be a countable base of neighborhoods of x. Construct increasing chains $(H_\alpha : \alpha < \omega_1)$ of subsets of X and $(\mathcal{B}(\alpha) : \alpha < \omega_1)$ of families of open sets in X such that:

(1) $|H_\alpha| \leq \mathfrak{c}$

(2) $\mathcal{B}(\alpha) = \bigcup\{\mathcal{B}_x : x \in \bigcup\{H_\beta : \beta < \alpha\}\}$.

(3) If for each $\gamma < \omega$, $\mathcal{W}_\gamma \subset \mathcal{B}(\alpha)$, $|\mathcal{W}_\gamma| \leq \omega$, and $X \setminus \bigcup\{\overline{\bigcup \mathcal{W}_\gamma} : \gamma < \omega\} \neq \emptyset$ then $H_\alpha \setminus \bigcup\{\overline{\bigcup \mathcal{W}_\gamma} : \gamma < \omega\} \neq \emptyset$.

The construction is like in Theorems 1 and 2. As soon as we have H_α for all α, put $H = \cup_{\alpha<\omega_1} H_\alpha$ and prove that $H = X$. $\square$

(1) Theorem 3 has a generalization: $|X| \leq 2^{c(X)\cdot\chi(X)}$ for every Hausdorff space X.

(2) The following queston by Arhangelskii remains open for more than 30 years: does every C.C.C. symmetrizable[7] space have cardinality $\leq \mathfrak{c}$?

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[3] Here you will need T_2 while the rest of the argument uses only T_1; however, we assume all spaces to be Tychonoff...

[4] This also shows that Theorem 2 and the previous remark do not have common generalization.

[5] Thus, for a compact X, $ic(X)$ is undefined; in this case we may for convenience say that $ic(X)$ is "greater than any cardinal". For a countably compact X, $ic(X) > \omega$, etc.

[6] Later we will consider also a Ramsey-theoretic proof of this theorem

[7] X is symmetrizable if there is a mapping $d : X^2 \to [0, \infty)$ such that (1) $d(x, y) = 0$ iff $x = y$, (2) $d(x, y) = d(y, x)$ for all x and y, and (3) $U \subset X$ is open iff for every $x \in U$, there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$ (where $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$); see Gruenhage's Handbook article. Symmetrizable spaces are close to first countable in some sense but do not have to be first-countable (can you construct an example?) Another long-standing question about symmtrizable spaces: do all regular symmetrizable spaces have countable pseudocharacter? (There is a paper of Gruenhage, Davis and Nyikos about this problem).

Other inequalities that can be proved by saturation method:[8]

(1) $|X| \leq 2^{l(X)\cdot\psi(X)\cdot t(X)}$ for Hausdorff X (Arhangelskii, see [Hodel] 4.6)
(2) $|X| \leq 2^{wl(X)\cdot\chi(X)}$ for a normal X (Bell, Ginsburg, Woods, see [Hodel], 4.13)[9]
(3) $|X| \leq 2^{hc(X)\cdot\psi(X)}$ for T_1 space X (Hajnal-Juhasz, see [Hodel], 4.9)[10]
(4) $|X| \leq 2^{e(X)\cdot\Psi(X)}$ for Hausdorff X[11] (see [Hodel], 9.5).
(5) Therefore[12] $|X| \leq 2^{hl(X)}$ for Hausdorff X (de Groot)
(6) $w(X) \leq \chi(X)^{c(X)}$ for regular X (compare with [Hodel], 6.2)
(7) $hd(X) \leq (hc(X))^+$ for compact X (Shapirovskii, see [Hodel], 7.17)

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[8] For reference only, I do not expect that you study all these with proofs by the next meeting

[9] Here $wl(X) = \min\{\tau :$ for every open cover $\mathcal{U}$ of X there is $\mathcal{F} \subset \mathcal{U}$ such that $|\mathcal{W}| \leq \tau$ and $\overline{\bigcup \mathcal{W}} = X\}$ is the weak Lindelöf number of X.

[10] This can also be proved using Ramsey-theoretic technique

[11] Moreover, $|\mathcal{K}(X)| \leq 2^{e(X)\cdot\Psi(X)}$ where $\mathcal{K}(X)$ is the family of all compact subsets of X and $\Psi(X) = \min\{\tau :$ for every closed $H \subset X$ there is a family of open sets $\mathcal{F}$ such that $|\mathcal{F}| \leq \tau$ and $\bigcap \mathcal{F} = H$

[12] Why?