# 7. Applications of set mappings

A *set mapping* on a set $S$ is a mapping $F$ that assigns to every $x \in S$ a (usually assumed to be non-empty) subset $F(x) \subset S$ so that $x \notin F(x)$. A set mapping $F$ is *of order* $\lambda$ if $|F(x)| < \lambda$ for all $x \in S$. A subset $A \subset S$ is *$F$-free* if $y \notin F(x)$ for all $x, y \in A$. A subset $A \subset S$ is *closed under* $F$ if $f(x) \subset A$ for every $x \in A$.

**Theorem 1.** (Hajnal, see [Juhasz] A3.5)) *Let $|S| = \kappa > \tau$ and let $F$ be a set mapping on $S$ of order $\tau$. Then $S$ contains an $F$-free subset of cardinality $\kappa$.*

*Sketch of proof:* Let us consider only the simple case when $\tau < \mathrm{cf}(\kappa)$ (this case is due to Lazar, see [Juhasz] A3.4). Assume that every $F$-free subset of $S$ has cardinality $< \kappa$. Then one can construct by induction for all $\alpha < \tau$ $F$-free susbsets $S_\alpha$ so that:

(1) $S_0$ is a maximal (with respect to inclusion) $F$-free subset of $S$, and

(2) for $0 < \alpha < \tau$, $S_\alpha$ is a maximal $F$-free subset of $S \setminus \bigcup_{\beta<\alpha} S_\beta$.

Put $S^* = \bigcup_{\alpha<\tau} S_\alpha$ and $S^{**} = S^* \cup \bigcup\{F(x) : x \in S^*\}$. Then $|S^{**}| < \kappa$. Pick $z \in S \setminus S^{**}$. Then $F(z)$ intersects $S_\alpha$ for each $\alpha < \tau$ (otherwise $S_\alpha$ would not be maximal). Since $S_\alpha$ are pairwise disjoint it follows that $|F(z)| \geq \tau$, a contradiction □

**Lemma 2.** (See 3.3 in [Juhasz]) *Let $X$ be a topological space, $|X| = \kappa > \tau \geq \omega$. Put $X_\tau = \{x \in X : \text{there is a neighborhood } U_x \ni x \text{ with } |U_x| < \tau\}$. If $|X_\tau| = \kappa$ then $X$ contains a discrete subspace of cardinality $\kappa$.*

*Proof:* Apply Hajnal's theorem to $S = X_\tau$ and $F(x) = (U_x \setminus \{x\}) \cap S$. □

Recall that spread, extent, etc, are defined in the terms of sup (unlike other cardinal functions, such as weight, density, etc. which are defined in the terms of min). This generates the sup vs. max problem (see Section 3 of [Juhasz]). Of course sup = max if sup is a successor cardinal. For limit cardinals, the situation is more delicate.

(1) (Easy) Give an example of a cardinal $\tau$ and a space $X$ with $e(X) = \tau$ such that $X$ does not have a closed discrete subspace of cardinality $\tau$.

(2) (More difficult) Give an example of a cardinal $\tau$ and a space $X$ with $s(X) = \tau$ such that $X$ does not have a discrete subspace of cardinality $\tau$.

Set functions technique shows that sometimes sup = max for singular cardinals.

**Theorem 3.** ([Juhasz], 3.3) *Let $s(X) = \kappa$ where $\mathrm{cf}(\kappa) = \omega$. Then $X$ contains a discrete subspace of cardinality $\kappa$.*[1]

*Sketch of proof:* Let $\kappa = \sum_{n<\omega} \kappa_n$ where $\kappa_n$s are regular and increasing with $n$. Then for every $n \in \omega$ there is a discrete set $D_n \subset X$ with $|D_n| = \kappa_n$. Put $X' = \bigcup_{n\in\omega} D_n$. □

**Theorem 4.** (Fodor, [1], see [3], Theorem 3.1.5) *Let $S$ be a set of cardinality $\kappa$ and let $F$ be a set mapping on $S$ of order $\tau < \kappa$. Then there exists a family $\mathcal{H}$ of $F$-free subsets such that $|\mathcal{H}| \leq \tau$ and $\mathcal{H}$ covers $S$.*

*In particular, if $S$ is a set of cardinality $\kappa$ and $F$ a set mapping on $S$ of order $\omega$, then there exists a countable family $\mathcal{H}$ of $F$-free subsets such that $\mathcal{H}$ covers $S$.*

I am skipping the proof. In case you are interested, I will post it separately.

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[1] Similar statements are true about $hd$ and $hl$.

**Theorem 5.** (Matveev, [2]) *The extent of a Tychonoff star-Lindelöf*[2] *space can be arbitrarily large.*

*Sketch of Proof:* Let $\tau$ be an infinite cardinal. We will construct a star-Lindelöf space $X$ such that $X$ contains a closed discrete subspace $X_1$ of cardinality $\tau$. For $\alpha < \tau$, let $z_\alpha$ be the point in $2^\tau$ such that only the $\alpha^{\text{th}}$ coordinate of $z_\alpha$ equals 1. Put $Z = \{z_\alpha : \alpha < \tau\}$. Let $\kappa$ be a cardinal such that $\mathrm{cf}(\kappa) > \tau$. Put $X = X_0 \cup X_1$ where $X_0 = 2^\tau \times \kappa$ and $X_1 = Z \times \{\kappa\}$; $X$ is considered as a subspace of $2^\tau \times (\kappa + 1)$. Obviously, $X_1$ is closed and discrete in $X$ and of cardinality $\tau$. So it remains to prove that $X$ is star-Lindelöf.

$X_0$ is countably compact and thus star-Lindelöf. It remains to show that $X_1$ is relatively star-Lindelöf in $X$. This will be done if we show that for every choice, for all $\alpha < \tau$, of neighborhoods $U_\alpha \ni \langle z_\alpha, \kappa\rangle$ there is a countable set $C \subset X$ which intersects each $U_\alpha$. Without loss of generality we may assume that $U_\alpha$s are of the form $U_\alpha = B_\alpha \times (\xi_\alpha, \kappa]) \cap X$ where $B_\alpha$ is an element of the standard base of $2^\tau$.

We claim that there is a countable $C_0 \subset 2^\tau$ which intersects all $B_\alpha$. Since $B_\alpha \ni z_\alpha$we may assume that there is a finite set $K_\alpha \subset \tau$ such that $B_\alpha = \{x \in 2^\tau : x(\alpha) = 1$ and $x(K_\alpha) = \{0\}$. Then $\alpha \mapsto K_\alpha$ defines a set mapping $F$ on $\tau$. By Fodor Theorem there is a countable family $\mathcal{H}$ of $F$-free sets such that $\bigcup \mathcal{H} = \tau$. For $H \in \mathcal{H}$ let $c_H$ be the indicator function of $H$. Put $C_0 = \{c_H : H \in \mathcal{H}\}$.

Now choose $\xi$ with $\kappa > \xi > \sup\{\xi_\alpha : \alpha < \tau\}$ and put $C = C_0 \times \{\xi\}$. $\square$[3]

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(1) Let $F$ be a set mapping on $\mathbb{R}$ such that for every $x$, $F(x)$ is a closed subset of $\mathbb{R}$. Show that (even if Hajnal's theorem does not help) there exists an F-free subset $A \subset \mathbb{R}$ such that $|A| = \mathfrak{c}$.[4] Moreover, $\mathbb{R}$ can be covered by countably many $F$-free sets. *Hint:* For every $x$, pick an element $U_x$ of the standard countable base of $\mathbb{R}$ such that $x \in U_x \subset \mathbb{R} \setminus F(x)$ and apply the Pigeon-Hole principle.

(2) Generalize the previous observation as wide as you can.

(3) In general say that a set mapping on a topological space $X$ is *closed* if for every $x \in X$, $F(x)$ is closed. Put $\mathcal{F}(X) = \sup\{\tau :$ for every closed set mapping $F$ on $X$ there is an $F$-free subset of cardinality $\geq \tau\}$. Thus, for example, $\mathcal{F}(\mathbb{R}) = \mathfrak{c}$. Calculate $\mathcal{F}(X)$ for those $X$ for which you think it might be interesting, or find general facts. For example (easy), is there $X$ with $\mathcal{F}(X) < |X|$? Is there $X$ with $\mathcal{F}(X) < |X|$ and such that every non-empty open set in $X$ has the same cardinality as $X$?

(4) (Bagemihl, see [3], p.70) Say that a set mapping defined on a topological space $X$ is *nowhere dense* if $F(x)$ is nowhere dense in $X$ for every $x \in X$. Every nowhere dense mapping $F$ defined on $\mathbb{R}$ has a $F$-free subset which is dense in $\mathbb{R}$.

(5) There is a nowhere dense set mapping $F$ defined on $\mathbb{Q}$ such that there is no dense in $\mathbb{Q}$ $F$-free subspace. *Hint:* enumerate $\mathbb{Q} = \{q_n : n \in \omega\}$ and put $F(q_n) = \{q_0, ..., q_{n-1}, q_{n+1}\}$.

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[2]See part 6 of these notes for the definition of star-Lindelöf

[3]There is an alternative proof by Gady Kozma: $Z$ defined like in the proof above is a closed discrete subspace of $2^\tau \setminus \overrightarrow{0}$ where $\overrightarrow{0}$ is the point in $2^\tau$ with all coordinates equal to zero. It remains to show that $2^\tau \setminus \overrightarrow{0}$ is star-Lindelöf.

[4]This is motivated by the example in the first lines of [Juhasz], Appendix 3

## References


[1] G. Fodor, *Proof of a conjecture of P. Erdös*, Acta Sci. Math. (Szeged) **14** (1952) 219-227.
[2] M. Matveev, *How weak is weak extent?*, Topol. and Appl. **119** (2002) 229-232.
[3] N.H. Williams, *Combinatorial Set Theory*, North-Holland, Amsterdam, 1977.